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A biconnected set in the plane

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Abstract

The purpose of this paper is to raise again the question of B. Knaster and C. Kuratowski as to whether there exists a biconnected set in the plane without a dispersion point. Assuming that Martin's Axiom holds, an example of such a space is constructed which has the additional property of being widely connected.

Keywords: Connected; Plane; Biconnected; Widely connected

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The purpose of this paper is to again raise an old, basic unsolved question, to fill a gap I left open forty years ago, and to make a somewhat more modern commentary on these problems than would have been made at that time.

Following [1] and [2], a topological space is called *connected* if it has more than one point and is not the union of two disjoint closed subspaces. A space is *biconnected* if it is connected but is not the union of two disjoint connected subspaces. A simple fact, proved in [2], is that a biconnected space cannot have two disjoint connected subspaces. A space is *widely connected* if it is connected and all of its connected subspaces are dense in it. A point x of a space X is called a *dispersion point of* X if X is connected but $X - \{x\}$ is totally disconnected.

In 1921 [1], Knaster and Kuratowski defined biconnected and gave a number of examples of biconnected sets in the plane all of which have dispersion points and asked:

Basic question. Does there exist a biconnected set in the plane without a dispersion point?

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In 1937 [2], Miller, assuming the Continuum Hypothesis, constructed a biconnected, widely connected set in the plane, thus partially answering the Knaster–Kuratowski question since a widely connected set has no dispersion point. Miller then raised the obvious question of the existence of a biconnected set in the plane without a dispersion point which is not widely connected.

In 1953 [3], I announced that, assuming the Continuum Hypothesis, the Miller question also had a yes answer. The 1953 proof was never published and all traces of it have been lost. However in *this* paper, I prove a slightly stronger:

Theorem. *If Martin's Axiom holds, then there is a biconnected set X in the plane with neither a dispersion point for any connected subset nor a widely connected subset.*

In 1958 [4], I disproved a conjecture of Erdös by showing that the Continuum Hypothesis allowed one to construct a connected set X in the plane, every connected subset of which differed from X by an, at most, countable set: the ultimate type of biconnected, widely connected set. Martin's Axiom would also suffice for such a construction as well as for a construction similar to Miller's.

The proof of the above Theorem is messier because the space must have no widely connected subset, but this part of the proof is straightforward and requires no special set theoretic assumptions. Widely connected restrictions on the basic Knaster–Kuratowski question are just red herrings which confuse the issue of our inability to answer the basic question absolutely. The problem with all of the constructions given so far is that they involve inductively selecting points while avoiding an increasingly large set of nowhere dense subsets from a Cantor set of composants for some indecomposable continuum in the plane. As the number of sets to be avoided approaches the power of the continuum, some set theoretic assumption is required. To answer the basic question (or any of its variations) absolutely, we need a different technique (or a model of Zermelo–Frankel set theory in which there is no example).

Proof of Theorem. Let E be the interior of the rectangle in the plane bounded by the lines $x = -1$, $x = 1$, $y = -3$, $y = 3$. Let D be the part of E between the lines $y = -1$ and $y = 1$. Construct an indecomposable continuum K in $\overline{E}$ of the most elementary type. That is K is the complement of an open canal M that winds densely through E with banks L_1 and L_2, each homeomorphic to $\mathbb{R}$ which are unions of horizontal and vertical intervals. Do the construction in such a way that there is no turning of the canal between the lines $y = -2$ and $y = 2$. Thus $\overline{D} \cap K$ is just the union of a Cantor set of compact vertical intervals from $y = -1$ to $y = 1$.

If $p \in K$, the *composant of K containing p* is the union of all proper subcontinua of K containing p. The composants of K are disjoint, connected, and σ-compact. The boundary of E is one composant of K; L_1 and L_2 are each composants of K. Let $\mathcal{L}$ be the set of all composants of K different from the boundary of E. Every $L \in \mathcal{L}$ is dense in K and, since L is σ-compact, $\overline{D} \cap L$ is the union of a countable dense subset of the

Cantor set of vertical intervals whose union is $\overline{D} \cap K$. To fill up $\overline{D} \cap K$ it clearly takes $\mathfrak{c}$ many composants (where $\mathfrak{c}$ denotes the cardinality of the continuum). Choose $L_3 \in \mathcal{L}$ different from L_1 and L_2 and let $\mathcal{L}' = \mathcal{L} - \{L_1, L_2, L_3\}$.

Let $\mathcal{O}$ denote the line $y = 0$ and let $\mathcal{Y}$ and $\mathcal{Z}$ be disjoint countable families of horizontal lines intersecting D but not including $\mathcal{O}$ such that $Y = \bigcup \mathcal{Y}$ and $Z = \bigcup \mathcal{Z}$ both intersect D in a dense set.

Let $\{M_n \mid n \in \omega\}$ be an indexing of the components of $D \cap M$. Each M_n is the interior of a rectangle having vertical edges in L_1 and L_2, respectively, and horizontal edges on the lines $y = -1$ and $y = 1$, respectively. For each $n \in \omega$ let $N_n = (\overline{M}_n \cap L_1 \cap Y) \cup \{\langle x, y\rangle \in M_n \cap Y \mid x \text{ is rational}\} \cup \{\langle x, y\rangle \in M_n - (Y \cup Z) \mid x \text{ is irrational}\}$. Clearly N_n is not connected since, for each $W \in \mathcal{Z}$, the points of N_n above and below W, respectively, are separated. But the experienced reader may recognize the relationship between N_n and the well known Knaster–Kuratowski example of a connected set with a dispersion point [1, p. 241] and realize that the *only* separations of N_n are of this type. We prove what we need of this as it comes up in our proof. Let $N = \bigcup_{n\in\omega} N_n$.

By induction, for each $\alpha < \mathfrak{c}$ we later choose an $x_\alpha \in D \cup K$ such that defining $X = N \cup (L_3 \cap Z) \cup \{x_\alpha \mid \alpha < \mathfrak{c}\}$ yields a space as desired in the theorem. Among other things this X is chosen to satisfy:

(0) X is a dense subset of $D \cup K$.

(1) For all $n \in \omega, X \cap \overline{M}_n = N_n$.

(2) $X \cap Z = L_3 \cap Z = X \cap L_3$.

(3) $X \cap L_1 \subset N$ and $X \cap L_2 = \emptyset$.

(4) $X \cap K \cap \mathcal{O} = \emptyset$.

(5) $X \cap L$ is countable for all $L \in \mathcal{L}$.

Before doing our inductive construction, let us observe a couple of consequences of these properties:

(∗) *If $S \subset X$ is connected, then S is dense in K.*

Proof. Otherwise S misses an open U intersecting K. Like M, L_2 partitions $E - U$ so that a component of $(E - U) - L_2$ can intersect at most one composant of K. Since $S \cap L_2 = \emptyset$ by (3), if p and q are in different composants of K, then p and q cannot both be in S which is connected. By (1), (3), and (5) then, the only points of K which could be in S are in L_1, so $S \subset N$. But since $N \cap Z = \emptyset$ and S is connected, S is a subset of some horizontal line W intersecting D but not in $\mathcal{Z}$. But, again by (1), $S \cap W$ is then either contained in a countable set in case $W \in \mathcal{Y}$ or in the points of W with irrational abscissae in case $W \notin \mathcal{Y}$. Both of these contradict S being connected. □

(∗∗) *No subset of X is widely connected.*

Proof. Suppose $S \subset X$ is connected. There is an $n \in \omega$ with $M_n \cap S \neq \emptyset$. (Otherwise, since $\mathcal{O}$ separates K and $X \cap K \cap \mathcal{O} = \emptyset$ by (4), S is not dense in K contrary to (∗).)

Let $T = S - M_n$. Then T is nondegenerate since $\overline{T} \supset K$, and $\overline{T} \not\supset S$ since $\overline{T} \cap M_n = \emptyset$. So S is not widely connected if T is connected in the usual way.

To prove the latter we get a contradiction to the assumption that there are disjoint open sets U and V in the plane, both intersecting T, whose union contains T.

Let I be the vertical edge of M_n that is in L_1. By (1), $(X \cap (\text{boundary } M_n)) \subset I$. Since $(U \cap V) = \emptyset$ and $(S - M_n) \subset (U \cup V)$, the only boundary points of $U - \overline{M}_n$ in S are in I. Hence there is $p \in (S \cap I \cap U)$ for, if $(S \cap I \cap U) = \emptyset$, then the boundary of $U - \overline{M}_n$ misses $S \cap (U - \overline{M}_n)$ and $S - (S \cap (U - \overline{M}_n))$ which are then separated and nonempty yielding a separation of the connected set S. Similarly there is $p' \in (S \cap I \cap V)$.

Since S is dense in K and $(X - \overline{D}) \subset \bigcup \mathcal{L}'$ by (2) and (3), there is $q \in (S - \overline{D}) \subset (\bigcup \mathcal{L}')$. Say $q \in U$. Since $p' \in (V \cap I)$ and p' is a limit of $K - \overline{M}_n$, there is an open $W \subset (V \cap D)$ with $W \cap K \neq \emptyset$ but $W \cap \overline{M}_n = \emptyset$.

Again using the fact that L_2 follows M, since p and q are in different composants of K, there is a compact $F \subset L_2$ such that p and q are in different components of $(E - W) - F$. Observe that since $p \in I \subset L_1$, $p \in (I \cup M_n)$ which is connected and missed $L_2 \cup W$. So $(I \cup M_n)$ is in the component of $(E - W) - F$ containing p. Let Q be the component of $U - F$ containing q. Remember that $U \subset E - W$ since $W \subset V$. Then Q is open, $q \in Q \cap S$, $p \in S - Q$, and the boundary of Q, which is contained in ((boundary of $U) \cup F) - (M_n \cup I)$ misses S. All of which is impossible since S is connected. $\square$

The construction of X

Index the set of all compact separating sets of $D \cup K$ as $\{F_\alpha \mid \alpha < \mathfrak{c}\}$. Index $\{C \subset (L_3 \cap Z) \mid C \text{ is dense in } W \cap K \text{ for all } W \in Z\}$ as $\{C_\alpha \mid \alpha < \mathfrak{c}\}$. The latter is possible because $L_3 \cap Z$ is countable. For each $\alpha < \mathfrak{c}$ we choose $x_\alpha \in F_\alpha$ such that either $x_\alpha \in N \cup (L_3 \cap Z)$ or $x_\alpha \in \bigcup \mathcal{L}' - (Z \cup \mathcal{O})$ with x_α not in the same composant of K as x_β for any $\beta < \alpha$. When we then define $X = N \cup (L_3 \cap Z) \cup \{x_\alpha \mid \alpha < \mathfrak{c}\}$ as mentioned earlier, we guarantee that X is connected and it gives us (0), (1), (2), (3), (4), and (5). By (∗), $K \subset \overline{S}$ for every connected $S \subset X$, so no connected subset of X has a dispersion point. By (∗∗), no subset of X is widely connected. To have our X satisfy the Theorem it remains to make our choice of the x_α in such a way that there are no two disjoint connected sets in X.

We use the C_α for this purpose. For a set A in the plane let $\pi(A)$ be the set of all 1st coordinates of points in A. Let $\mathcal{G}_\alpha$ be the set of all graphs G of continuous, nondecreasing functions from $(-1, 1)$ onto $(-1, 1)$ such that:

(i) $(G \cap M) \subset Z$,
(ii) $(G \cap L_3 \cap Z) \subset C_\alpha$,
(iii) $\pi(G \cap C_\alpha)$ is dense in $\pi(D \cap K)$.

In the same inductive process used to choose x_α we choose $G_\alpha \in \mathcal{G}_\alpha$ having the property that $(G_\alpha \cap \{x_\beta \mid \beta < \alpha\}) \subset C_\alpha$. Then we choose x_α as above with $x_\alpha \notin (G_\beta - C_\beta)$ for any $\beta \leqslant \alpha$.

Suppose S is a connected subset of X. If $S \cap W$ is not dense in $K \cap W$ for some $W \in \mathcal{Z}$, there is a nontrivial closed interval $[a, b]$ of W with a and b in L_2 such that

$[a,b] \cap S = \emptyset$. But if $[a,b]'$ is the interval of L_2 from a to b, then $[a,b] \cup [a,b]'$ separates K in the plane and misses S. Since, by $(*)$, S is dense in K, we have a contradiction. Thus, since $(S \cap Z) \subset (L_3 \cap Z)$, $(S \cap L_3 \cap Z) = C_\alpha$ for some $\alpha < \mathfrak{c}$. But, as before, a compact subset of $G_\alpha \cup L_2$ separates K and its intersection with X is contained in C_α. So there can be no connected subset of X disjoint from S since S contains C_α and by $(*)$, every connected subset of X is dense in K.

It remains to show that the inductive construction can actually be made. Assume $\alpha < \mathfrak{c}$ and that $\{x_\beta \mid \beta < \alpha\}$ and $\{G_\beta \mid \beta < \alpha\}$ have been chosen. Our inductive hypotheses are that, for all $\beta < \alpha$, $x_\beta \notin (Z - L_3)$ and $G_\beta \in \mathcal{G}_\beta$.

I. *We choose $G_\alpha \in \mathcal{G}_\alpha$ with $(G_\alpha \cap \{x_\beta \mid \beta < \alpha\}) \subset C_\alpha$.*

To prove this is possible we find a subfamily $\mathcal{G}$ of $\mathcal{G}_\alpha$ of cardinality $\mathfrak{c}$ such that $(G \cap G') \subset Z$ for all $G \neq G'$ in $\mathcal{G}$. If $\beta < \alpha$ and $x_\beta \in Z$, then $x_\beta \in L_3 \cap Z$ by hypothesis and, since $(G \cap L_3 \cap Z) \subset C_\alpha$ for each $G \in \mathcal{G}_\alpha$, if $x_\beta \notin C_\alpha$, x_β can belong to at most one $G \in \mathcal{G}_\alpha$. Since there are $< \mathfrak{c}$ many x_β and $\mathfrak{c}$ many G in $\mathcal{G}$, there is some $G \in \mathcal{G}$ with $G \cap \{x_\beta \mid \beta < \alpha\} \subset C_\alpha$.

Construction of $\mathcal{G}$

Let $Z = \{Z_n \mid n \in \omega\}$.

Let $\mathcal{T}_n$ be the set of all maximal horizontal closed intervals contained in $\overline{M}_n \cap Z$ and $\mathcal{T} = \bigcup_{n\in\omega} \mathcal{T}_n$.

Let $\mathcal{A}$ be the family of all rectangle interiors in D having horizontal and vertical edges and having upper right and lower left corners in C_α. For $A \in \mathcal{A}$, let $\Delta(A)$ be the diagonal of A from its lower left to its upper right corner (not including the corners).

We call $\langle \mathcal{P}, \leqslant \rangle$ a *chain* if:

(c) $\mathcal{P} \subset (\mathcal{A} \cup \mathcal{T}_W \cup C_\alpha)$.

(cc) $\leqslant$ is a total order on $\mathcal{P}$ where $A < B$ in $\langle \mathcal{P}, \leqslant \rangle$ implies A is both below and to the left of B.

(ccc) $\bigcup(\mathcal{P} \cap (\mathcal{T} \cup C_\alpha)) \cup \bigcup\{\Delta(A) \mid A \in (\mathcal{P} \cap \mathcal{A})\}$ is the graph of a continuous function from $(-1,1)$ onto $(-1,1)$.

Observe that since the elements of $\mathcal{T}$ contain their own end points, no two elements of $\mathcal{T}$ can abut in a chain. However, if $A \in (\mathcal{A} \cap \mathcal{P})$, the lower left and upper right corners of A must be members of $\mathcal{P}$ in their own right and, say the lower left corner of A must, of necessity, be the upper right corner of some other member of $\mathcal{A} \cap \mathcal{P}$. If p and q are points of D (or $\langle -1,-1 \rangle$ or $\langle 1,1 \rangle$) with p to the left of and below q, there are many ways to select an infinite string, as narrow, as one may wish, of members of $\mathcal{A}$, totally ordered by $\leqslant$, whose diagonals together with their lower left and upper right corners, form an arc from p to q. Building chains is as easy as building arcs.

By induction, for all $n \in \omega$ and $f : n \to 2$ we choose a chain $\mathcal{P}_f$ as follows. If $n = 0, f = \emptyset$ and we just let $\mathcal{P}_f$ be any chain such that $\mathcal{P}$ contains both a term of $\mathcal{T}_0$ and a term contained in Z_0.

Suppose $\mathcal{P}_f$ for $f:(n-1)\to 2$ has been defined and f_0 and f_1 are the functions extending f to n with $f_0(n-1)=0$ and $f_1(n-1)=1$. We choose $\mathcal{P}_{f_0}$ and $\mathcal{P}_{f_1}$ to be chains containing $\mathcal{P}_f-\mathcal{A}$, a term of $\mathcal{T}_n$, and a term (either from C_α or $\mathcal{T}$) contained in Z_n such that, for each $A\in\mathcal{P}_f\cap\mathcal{A}$, $(A\cap(\overline{\bigcup\mathcal{P}_{f_0}})\cap(\overline{\bigcup\mathcal{P}_{f_1}}))=\emptyset$.

If $g:\omega\to 2$, let

$$G_g=\bigcap_{n\in\omega}\Big\{\bigcup\mathcal{P}_f \mid f=g\restriction n\Big\}.$$

For every $n\in\omega$ there is a unique term T_n of $\mathcal{T}_n$ in G_g, so (i) $(G_g\cap M)\subset Z$. For every n there is a term T of G_g contained in Z_n and $(G_g\cap Z_n)=T$. If $T\in\mathcal{T}$, $(T\cap L_3)=\emptyset$, otherwise $T\in C_\alpha$; in any case, (ii) $(G_g\cap L_3\cap Z)\subset C_\alpha$. For each n, one of the endpoints of T_n, say t_n, is in L_1. Let $f=g\restriction n$. Since $T_n\in\mathcal{P}_f$ and $\pi(\bigcup\mathcal{P}_f)=(-1,1)$ and no open interval of $(-1,1)$ containing $\pi(t_n)$ can be filled by $\pi(\bigcup(\mathcal{T}_n\cap\mathcal{P}_f))$, $\pi(t_n)$ is a limit of $\{\pi(x)\mid x\in(C_\alpha\cap\mathcal{P}_f)\}$. Since $\pi(D\cap L_1)$ is dense in $\pi(D\cap K)$ and $\{\pi(t_n)\mid n\in\omega\}=\pi(D\cap L_1)$, (iii) $\pi(G_g\cap C_\alpha)$ is dense in $\pi(D\cap K)$. Hence $G_g\in\mathcal{G}_\alpha$.

Also if $g\neq g'$ in 2^ω, $(G_g\cap G_{g'})\subset(\mathcal{T}\cup C'_\alpha)\subset Z$. Thus $\mathcal{G}=\{G_g\mid g\in 2^\omega\}$ is as desired. □

II. *We choose* x_α.

First we need some definitions and a lemma.

There are a compact F and disjoint open U and V in the plane both intersecting $D\cup K$ such that $(U\cup V\cup F)\supset(D\cup K)$, $\overline{U}-U=\overline{V}-V=F$, and $F\cap(D\cup K)\subset F_\alpha$.

Lemma. *Suppose* $n\in\omega$ *and* $F\cap N_n=\emptyset$.

(See (1).) Define J_U to be the union of all open subintervals J of $(-1,1)$ such that $M_n\cap(\mathbb{R}\times J)\subset U$; define J_V similarly. *Then* $\mathbb{R}\times(J_U\cup J_V)$ *covers* M_n-Z.

Proof. If $p\in(\overline{M}_n\cap L_1\cap Y)$, then $p\in N_n$ so $p\in U$ or $p\in V$. Say $p\in U$. Thus there are open intervals J_1 and J_2 of $(-1,1)$ with p being the center point of $(J_1\times J_2)\subset U$. We claim that actually $(M_n\cap(\mathbb{R}\times J_2))\subset U$. Otherwise there are open intervals J_3 and J_4 of $\mathbb{R}$ with $J_2\cap J_4\neq\emptyset$ and $(J_3\times J_4)\subset(V\cap M_n)$. Let J_5 be the open interval of $\mathbb{R}$ between the midpoints of J_1 and J_3; and let $J_6=J_2\cap J_4$. Observe that $(J_5\times J_6)\subset M_n$. For each rational $a\in J_5$, $\{b\in J_6\mid\langle a,b\rangle\in F\}$ is nowhere dense in J_6 since $\{b\in J_6\mid\langle a,b\rangle\in Y\}$ is dense in J_6 and $\{\langle a,b\rangle\in Y\mid b\in J_6\}\subset(X\cap M_n)$ which misses F. Thus by the Baire category theorem there is a $b\in J_6$ such that $\langle a,b\rangle\notin F$ for any rational $a\in J_5$ and the line $y=b$ is not in $\mathcal{Y}$ or $\mathcal{Z}$ which are countable. Since $\{\langle a,b\rangle\in J_5\times J_6\mid a \text{ is irrational}\}\subset N_n$, $J_5\times\{b\}$ is contained in one of U and V contradicting the fact that one end of the interval $J_5\times\{b\}$ is in U and the other in V. Thus we have proved that $(M_n\cap(\mathbb{R}\times J_2))\subset U$ and $J_2\subset J_U$.

Since Y is dense in M_n, $J_U\cup J_V$ is dense in $(-1,1)$. If B is a line $y=b$ for some $b\notin J_U\cup J_V$, then $(B\cap M_n)\subset(\overline{U}\cap\overline{V})$ and therefore $(B\cap N_n)=\emptyset$. So $B\in\mathcal{Z}$. □

Returning to our choice of x_α, let $\mathcal{R} = \{L \in \mathcal{L}' \mid x_\beta \notin L \text{ for any } \beta < \alpha\}$ and $R = \bigcup \mathcal{R}$.

We choose $x_\alpha \in F$ from one of the following sets.

(j) $N \cup (L_3 \cap Z)$.

(jj) $R - D$.

(jjj) $(R \cap D) - (Z \cup \mathcal{O} \cup \bigcup_{\beta \leqslant \alpha} G_\beta)$.

Our theorem is proved when we prove this is possible. So assume the intersection of F with all of these sets is empty.

Choose $L \in \mathcal{R}$. Since $F \cap (R - D) = \emptyset$, $F \cap (L - D) = \emptyset$ and every component of $L - D$ is contained in just one of U and V.

Claim. *There are $a \in [-1, 1)$ and $x \in (-1, 1)$ such that $\langle x, 1\rangle \in L$ and one of $\langle x, 1\rangle$ and $\langle x, a\rangle$ is in U and the other in V.*

Proof. Assume the claim is false. We have, in particular, that if $\langle x, 1\rangle \in L$, then $\langle x, 1\rangle$ and $\langle x, -1\rangle$ are either both in U or both in V. Thus $L = L_U \cup L_V$ where $L_U = \bigcup\{$components of $L - D$ contained in $U\} \cup \bigcup\{$components of $L \cap D$ whose end points are contained in $U\}$ and L_V is similarly defined.

Let us show that one of L_U and L_V is empty. Remember that L is connected and $L = L_U \cup L_V$. Suppose $\langle x, y\rangle \in L_U$. If $\langle x, y\rangle \notin \overline{D}$ then $\langle x, y\rangle \in U - \overline{D}$ so $\langle x, y\rangle \notin \overline{L}_V$. If $\langle x, y\rangle \in \overline{D}$ there are open intervals H, H_1, and H_2 in R with $\langle x, 1\rangle \in (H \times H_1) \subset U$ and $\langle x, -1\rangle \in (H \times H_2) \subset U$ and $\langle x, y\rangle \in H \times (H_1 \cup H_2 \cup (-1, 1))$ which can contain no point of L_V. Hence $L_U \cap \overline{L}_V = \emptyset$ and similarly $L_V \cap \overline{L}_U = \emptyset$. So one of L_U and L_V is empty. Say $L = L_U$.

Since $(L - D) \subset U$ and L is dense in K, $(V \cap K) \subset D$. If $\langle x, y\rangle \in L \cap D$, $\langle x, 1\rangle \in U$ and, by the falseness of our claim $\langle x, y\rangle \notin V$. So $L \subset \overline{U}$ and $K \subset \overline{L} \subset \overline{U}$. Thus $(X \cap V) \subset (\bigcup_{n\in\omega} M_n)$. Choose $n \in \omega$ with $V \cap N_n \neq \emptyset$. Then, by our lemma, there is a nonempty open J_V in $(-1, 1)$ with $M_n \cap (\mathbb{R} \times J_V) \subset V$. There is $p \in \overline{M}_n \cap L_1 \cap Y$ with its ordinate in J_V. Thus $p \in \overline{V}$ and $p \in K \cap \overline{L} \subset \overline{U}$ so $p \in F$. This contradicts $F \cap N_n = \emptyset$. □

Our claim having been proved we can choose open intervals H_1 and H_2 in $\mathbb{R}$ such that $B = (H_1 \times H_2) \subset D$, $B \cap K \neq \emptyset$, and B has one horizontal edge in U and the other in V.

Choose B so it lies entirely above $\mathcal{O}$ or below $\mathcal{O}$. To see that this is possible, suppose $\mathcal{O}$ intersects B. Since $B \cap K \neq \emptyset$, we can choose an n with $\pi(\overline{M}_n) \subset H_1$. Define J_U and J_V for this n as in our lemma. Either $0 \in J_U$ or $0 \in J_V$, say $0 \in J_V$. Let b be the end of H_2 with $(H_1 \times \{b\}) \subset U$. Choose $\langle x, y\rangle \in (\overline{M}_n \cap L_1 \cap Y)$ with $y \in J_V \cap (0, b)$. Since $\langle x, y\rangle \in \overline{V}$ and $\langle x, y\rangle \in N \subset$ (j) which misses F, $\langle x, y\rangle \in V$ and $\langle x, y\rangle \in (H_3 \times H_4) \subset V$. If $H_1' = H_1 \cap H_3$ and $H_2' = (y, b)$, then $H_1' \times H_2'$ misses $\mathcal{O}$ and has the other properties desired for B.

To ensure that $\pi(\overline{B} \cap K)$ is a Cantor set, choose B so its vertical edges lie in $K - (L_1 \cup L_2)$. This is clearly possible by narrowing some previously chosen H_1.

If $\beta \leqslant \alpha$, then $\pi(G_\beta \cap C_\beta)$ is dense in $\pi(D \cap K)$ and $\pi(G_\beta)$ is one-to-one. Since $F \cap C_\beta = \emptyset$ because $C_\beta \subset (L_3 \cap Z) \subset$ (j), $\pi(F \cap G_\beta \cap K)$ is nowhere dense in $\pi(\overline{B} \cap K)$. Similarly, if $W \in \mathcal{Z}$ which is countable, $W \cap L_3$ is dense in $W \cap K$ and

$$(F \cap L_3 \cap Z) = \emptyset$$

since $(L_3 \cap Z) \subset$ (j); so $\pi(F \cap W)$ is nowhere dense in $\pi(\overline{B} \cap K)$. Since $\mathcal{L} - \mathcal{R}$ has cardinality $< \mathfrak{c}$ and, for any $L \in \mathcal{L}$, $\pi(D \cap L)$ is countable, the cardinality of $\pi(D \cap \bigcup(\mathcal{L} - \mathcal{R}))$ is $< \mathfrak{c}$.

We now assume that *a Cantor set (namely $\pi(\overline{B} \cap K)$) is not the union of $< \mathfrak{c}$ many nowhere dense sets*. Thus Martin's Axiom implies that there is an $a \in \pi(\overline{B} \cap R)$ such that $a \notin \pi(F \cap G_\beta \cap K)$ for any $\beta \leqslant \alpha$ and $a \notin \pi(F \cap W)$ for any $W \in \mathcal{Z}$. If $H_2 = (b_1, b_2)$, one of $\langle a, b_1 \rangle$ and $\langle a, b_2 \rangle$ is in U and the other is in V. So there must be some $b \in H_2$ with $\langle a, b \rangle \in F$. Since

$$\langle a, b \rangle \in (R \cap D) - \left(Z \cup \mathcal{O} \cup \bigcup_{\beta \leqslant \alpha} G_\beta \right) = \text{(jjj)},$$

we have a contradiction to $F \cap \text{(jjj)} = \emptyset$.

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